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# OPTIMAL LOWER BOUNDS ON LOCAL STRESS INSIDE RANDOM MEDIA*

BACIM ALALI† AND ROBERT LIPTON‡

**Abstract.** A methodology is presented for bounding the higher $L^p$ norms, $2 \leq p \leq \infty$, of the local stress inside random media. We present optimal lower bounds that are given in terms of the applied loading and volume fractions for random two phase composites. These bounds provide a means to measure load transfer across length scales relating the excursions of the local fields to applied loads. These results deliver tight upper bounds on the macroscopic strength domains for statistically defined heterogeneous media.

**Key words.** random heterogeneous materials, failure criteria, stress concentrations, elastic–perfectly plastic

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## 1. Introduction.

Many structures are hierarchical in nature and are made up of substructures distributed across several length scales. Examples include aircraft wings made from fiber reinforced laminates, bridges made from steel reinforced concrete, and naturally occurring structures like bone. The applied load can be greatly amplified by the local microstructure and can result in local stress concentrations; see, for example, [21]. The presence of large local stress precedes the appearance of nonlinear phenomena such as fracture and yielding [2]. Thus it is crucial to quantify load transfer between length scales when considering failure initiation inside multiscale heterogeneous media. Any improvement in our understanding of failure initiation inside multiphase media has the potential to reduce the high cost involved in the development of advanced composite architectures for aerospace and infrastructure [4]. In this paper we present a new method for quantifying load transfer between length scales when the substructure or microstructure is known only in a statistical sense. New tools are provided for teasing out relationships that connect the local stress field to applied macroscopic loads. These relationships provide explicit criteria on the applied loads that are necessary for failure initiation inside statistically defined heterogeneous media.

Over the last century major strides have been made in the characterization of effective constitutive laws relating average fluxes and gradients inside heterogeneous media; see, for example, [13, 31, 32, 35, 42, 46]. However, knowledge of effective properties alone is not sufficient for the quantitative description of load transfer across length scales. Suitable mathematical quantities need to be invoked that are sensitive to the presence of zones of high field values inside heterogeneous media. Such quantities include the $L^p$ norms of the deviatoric and hydrostatic components of the local stress and strain. Higher $L^p$ norms of local fields are often used to describe phenomena re-

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http://www.siam.org/journals/siap/70-4/74496.html
†Department of Mathematics, University of Utah, Salt Lake City, UT 84112 (alali@math.utah.edu).
‡Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803 (lipton@math.lsu.edu).

lated to failure initiation inside heterogeneous media. In the applications the $L^\infty$ norm of the local stress is used to describe the strength domain for both elastic–perfectly plastic, periodic fiber reinforced composites [9] and for random, rigid–perfectly plastic composites and polycrystals; see, for example, [45, 43, 37, 40, 41, 6, 8, 22, 36]. For $p < \infty$ the $L^p$ norm of the local Von Mises stress is used in the description of failure probabilities; see [2, 21, 19].

This paper examines the local stress fields inside statistically homogeneous two phase random elastic media. The analysis focuses on the linear elastic regime. Here the random medium is characterized by a spatially varying elasticity tensor that provides the local constitutive law relating the local stress and strain. Most often one does not have access to the underlying probability measure describing the random elasticity tensor field, and instead one must make use of partial statistical information describing the random microstructure. In this paper we address the case when only the volume fractions of the two materials are known. New methods are presented that deliver explicit lower bounds on the $L^p$ norms of the local stress inside two phase heterogeneous random media. The bounds are given in terms of the applied loads, volume fractions, and elastic constants of the two materials. Several new lower bounds are presented for a ladder of progressively more complicated macroscopic load cases and are valid for the full range $2 \leq p \leq \infty$. These bounds are shown to be optimal and provide a means to measure load transfer across length scales relating the excursions of the local stress to the applied macroscopic loading. Here we have focused on lower bounds since volume constraints alone do not preclude the existence of microstructures with rough interfaces for which the $L^p$ norms of local fields are divergent; see [33, 5] and also [23]. The methods developed here can be used to obtain optimal lower bounds for local strain fields inside random heterogeneous media [1].

The results presented in this paper provide new quantitative tools for the study of failure initiation inside random heterogeneous media. For a given realization of the random medium, the theory of failure initiation posits that failure is initiated when certain rotational invariants of the local elastic stress exceed threshold values [21]. An example is an elastic–perfectly plastic material. Here the material deforms elastically up to some threshold value and then yields, undergoing plastic or irreversible deformation [18]. Typical stress invariants used to describe failure include the local hydrostatic stress component $\sigma^H$, which measures the hydrostatic force acting inside the material, and the Von Mises equivalent stress $\sigma^V$, which measures the local shearing forces acting inside a material [21]. Various combinations of these two invariants are considered in the strength of composites literature; see [3] and [47].

To fix ideas we introduce the macroscopic strength domain associated with the local Von Mises stress $\sigma^V$ for two phase statistically homogeneous random elastic media. Here we suppose that only the volume fractions $\theta_1$ and $\theta_2$ of the two elastic materials are known. The macroscopic strength domain $K^{Safe}$ is defined to be the set of applied constant stresses $\overline{\sigma}$ such that $\sigma^V$ lies below the failure threshold inside each component material almost surely for every microstructure realization of the random medium with prescribed volume fractions $\theta_1$ and $\theta_2$. An upper bound on the macroscopic strength domain is defined to be the set $\overline{K}$ of constant stresses such that if $\overline{\sigma}$ lies outside $\overline{K}$, then $\sigma^V$ has attained the threshold on some subset inside one of the component materials for every microstructure composed of materials one and two with prescribed volume fractions $\theta_1$ and $\theta_2$, so

$$K^{Safe} \subset \overline{K}. \tag{1.1}$$

In section 4 we apply the lower bounds on local fields to obtain explicit tight upper bounds on the macroscopic strength domains for statistically homogeneous random media. These results provide new optimal upper bounds on the strength domain for some prototypical examples of *elastic–perfectly plastic* random heterogeneous media. In this context we point out the substantial mathematical literature and associated theory characterizing the strength domains of heterogeneous media made from *rigid-perfectly plastic* materials; see, for example, [45, 43, 37, 40, 41, 6, 8, 22, 36]. Unlike an elastic–perfectly plastic material, a rigid–perfectly plastic material does not deform until yield occurs. For rigid–perfectly plastic materials the local stress satisfies only the equilibrium equation $\operatorname{div} \sigma = 0$ until the yield limit is reached. This is distinct from the elastic–perfectly plastic model where the stress also satisfies a constitutive law relating it to the local elastic strain.

Earlier work provides optimal lower bounds on local fields for random media subjected to applied constant hydrostatic stress and strain and for applied constant electric fields; see [27, 28, 26]. Those efforts deliver optimal lower bounds on the $L^p$ norms for the hydrostatic components of local stress and strain fields as well as the magnitude of the local electric field for all $p$ in the range $2 \leq p \leq \infty$. Other work examines the stress field around a single simply connected stiff inclusion subjected to a remote constant stress at infinity [49] and provides optimal lower bounds for the supremum of the maximum principal stress. The work presented in [12] provides an optimal lower bound on the supremum of the maximum principal stress for two-dimensional periodic composites consisting of a single simply connected stiff inclusion in the period cell. The recent work of [16] builds on the earlier work of [27, 28] and develops new lower bounds on the $L^p$ norm of the local stress and strain fields inside statistically isotropic two phase elastic composites. However, to date those bounds have been shown to be optimal for $p = 2$ (see [16]); their optimality for $p > 2$ remains to be seen. Optimal upper and lower bounds on the $L^2$ norm of local gradient fields are established using integral representation formulas in [29].

The paper is organized as follows. In the next section we present the elastic boundary value problem for heterogeneous media. Section 3 lists lower bounds for a ladder of load cases of increasing generality. The microstructures that support local fields that attain the lower bounds are introduced and discussed in this section. Upper bounds on the strength domains for random media are displayed in section 4. The lower bounds on the local stress are derived in section 5. Their attainability is demonstrated in section 6.

The hydrostatic and deviatoric components of local stress fields are defined below for future reference. We denote generic stress or strain tensor fields by $\psi(\mathbf{x})$ and $\eta(\mathbf{x})$. Contractions of two such fields $\psi$ and $\eta$ are defined by $\psi : \eta = \psi_{ij}\eta_{ij}$ and $|\psi|^2 = \psi : \psi$, where repeated indices indicate summation. Products of fourth order tensors $C$ and stress or strain tensors $\psi$ are written as $C\psi$ and are given by $[C\psi]_{ij} = C_{ijkl}\psi_{kl}$, and products of stresses or strains $\eta$ with vectors $\mathbf{v}$ are given by $[\eta \mathbf{v}]_i = \eta_{ij} v_j$. The fourth order identity map on the space of stresses or strains is denoted by $\mathbf{I}$ and $\mathbf{I}_{ijkl} = 1/2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. The projection onto the hydrostatic part of $\psi(\mathbf{x})$ is denoted by $\mathbf{\Pi}^H$ and is given explicitly by

$$\mathbf{\Pi}^H_{ijkl} = \frac{1}{d}\delta_{ij}\delta_{kl} \quad \text{and} \quad \mathbf{\Pi}^H \psi(\mathbf{x}) = \frac{\operatorname{tr} \psi(\mathbf{x})}{d} I. \tag{1.2}$$

The projection onto the deviatoric part of $\psi(\mathbf{x})$ is denoted by $\mathbf{\Pi}^D$ and $\mathbf{I} = \mathbf{\Pi}^H + \mathbf{\Pi}^D$ with $\mathbf{\Pi}^D\mathbf{\Pi}^H = \mathbf{\Pi}^H\mathbf{\Pi}^D = 0$. When $\psi(\mathbf{x})$ represents the local stress tensor, the well-known Von Mises equivalent stress is given by $|\mathbf{\Pi}^D\psi(\mathbf{x})|$. For completeness we introduce the following notation. The rank one matrix formed by taking the outer product of two unit vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\mathbf{a}\otimes\mathbf{b}$ with elements $(\mathbf{a}\otimes\mathbf{b})_{ij} = a_i b_j$. The symmetric part of this matrix is denoted by $\mathbf{a}\odot\mathbf{b}$ with elements $(\mathbf{a}\odot\mathbf{b})_{ij} = (a_i b_j + a_j b_i)/2$.

**2. Stress and strain fields inside stationary random heterogeneous media.** We present the equilibrium equations and constitutive laws used to describe the behavior of local stress and strain fields inside statistically homogeneous random heterogeneous materials [7, 20, 38, 46]; see also [32]. Every realization $\omega$ of the heterogeneous medium occupies $\mathbf{R}^d$, $d = 2, 3$, and is composed of two elastically isotropic materials with elasticity tensors denoted by $C^1$ and $C^2$. The bulk and shear moduli of materials one and two are denoted by $\kappa_1$ and $\mu_1$, and $\kappa_2$ and $\mu_2$, respectively. The isotropic elasticity tensor associated with each component material is given by

$$C^i = 2\mu_i \mathbf{\Pi}^D + d\kappa_i \mathbf{\Pi}^H \qquad \text{for } i = 1, 2, \tag{2.1}$$

where $d = 2$ for planar elastic problems and $d = 3$ for the three-dimensional problem.

Each realization of the random medium is specified by the indicator functions of phases one and two, denoted by $\chi_1(\mathbf{x},\omega)$ and $\chi_2(\mathbf{x},\omega)$. For a given realization $\chi_1(\mathbf{x},\omega)$ takes the value 1 in phase one and zero outside and $\chi_2(\mathbf{x},\omega) = 1 - \chi_1(\mathbf{x},\omega)$. The elastic tensor associated with the two phase medium is denoted by $C(\mathbf{x},\omega)$ and $C(\mathbf{x},\omega) = \chi_1(\mathbf{x},\omega)C^1 + \chi_2(\mathbf{x},\omega)C^2$. Here the index $\omega$ belongs to the sample space $\Omega$, and the associated probability measure $\mathcal{P}$ is defined over $\Omega$. For the class of statistically homogeneous or strictly spatially stationary and ergodic random media the joint distribution of the sets of indicator functions (for $n = 1, 2, \ldots$),

$$\chi_1(\mathbf{x}_1,\omega), \chi_1(\mathbf{x}_2,\omega), \chi_1(\mathbf{x}_3,\omega), \ldots, \chi_1(\mathbf{x}_n,\omega), \tag{2.2}$$

is invariant under all translations, and the ensemble averages of $\chi_1$ coincide with the mean value $\langle\chi_1\rangle$ defined as the limit of volume averages taken over progressively larger volumes [7, 20, 46]. The volume (area) fractions of phases one and two are given by the mean values

$$\theta_1 = \langle\chi_1\rangle \quad \text{and} \quad \theta_2 = \langle\chi_2\rangle, \tag{2.3}$$

and $\theta_1 + \theta_2 = 1$.

In what follows we suppress the variable $\omega$ when describing local stress and strain fields associated with a fixed microstructure realization. A constant "macroscopic" stress $\overline{\sigma}$ is imposed on the heterogeneous material. The local stress is expressed as the sum of a stationary, ergodic, mean zero fluctuation $\hat{\sigma}$ and $\overline{\sigma}$, i.e., $\sigma(\mathbf{x}) = \overline{\sigma} + \hat{\sigma}(\mathbf{x})$, with $\langle\hat{\sigma}\rangle = 0$. The equation of elastic equilibrium inside each phase is given by

$$\operatorname{div}\sigma = 0. \tag{2.4}$$

The local elastic strain $\epsilon(\mathbf{x})$ is related to the local stress through the constitutive law

$$\sigma(\mathbf{x}) = C(\mathbf{x})\epsilon(\mathbf{x}), \tag{2.5}$$

and the local elastic strain field $\epsilon(\mathbf{x})$ is written in the form

$$\epsilon(\mathbf{x}) = \overline{\epsilon} + \hat{\epsilon}(\mathbf{x}), \tag{2.6}$$

where $\hat{\epsilon}$ is a stationary, ergodic, mean zero strain fluctuation. The strain fluctuation is given in terms of the displacement field $\hat{\mathbf{u}}$ with $\hat{\epsilon}_{ij}(\mathbf{x}) = (\partial_j \hat{u}_i(\mathbf{x}) + \partial_i \hat{u}_j(\mathbf{x}))/2$. The traction at an interface with unit normal vector $\mathbf{n}$ pointing into material two is denoted by the product $\sigma\mathbf{n}$ and is the vector with components given by $[\sigma\mathbf{n}]_i = \sigma_{ij}n_j$. Perfect contact between the component materials is assumed; thus both the displacement $\hat{\mathbf{u}}$ and traction $\sigma\mathbf{n}$ are continuous across the two phase interface, i.e.,

$$\hat{\mathbf{u}}_{|_1} = \hat{\mathbf{u}}_{|_2}, \tag{2.7}$$

$$\sigma_{|_1}\mathbf{n} = \sigma_{|_2}\mathbf{n}. \tag{2.8}$$

Here the subscripts indicate the side of the interface that the displacement and traction fields are evaluated on. The existence and uniqueness of the required stationary ergodic stress and strain fluctuations $\hat{\sigma}$, $\hat{\epsilon}$ are well known and can be found in [7, 20, 46].

The effective "macroscopic" constitutive law for the random heterogeneous medium is given by the constant effective elasticity tensor $C^e$ [7, 20, 32, 46], relating the average imposed macroscopic stress $\overline{\sigma}$ to the average strain $\overline{\epsilon}$,

$$\overline{\sigma}_{ij} = C^e_{ijkl}\overline{\epsilon}_{kl}. \tag{2.9}$$

In what follows, bounds are derived on the moments of the local stress $\sigma$ defined on $\mathbf{R}^d$. Here the moments of a field $q$ are defined to be $\langle |q|^r\rangle^{1/r}$. For future reference we remind the reader that $\lim_{r\to\infty}\langle |q|^r\rangle^{1/r}$ is the same as the $\|q\|_\infty$ norm more commonly defined as the essential supremum of $q$; see [25].

## 3. Optimal lower bounds on the local stress inside random composites.

In this section we list new optimal lower bounds on the local stress for a ladder of progressively more general sets of imposed macroscopic stress. As we progress to more general load cases we will apply additional hypotheses on the shear and bulk moduli of the constituent materials. In this section we display lower bounds for the following applied macroscopic load cases: (1) lower bounds on the full local stress for imposed hydrostatic stresses; (2) lower bounds on the full local stress inside the material, with larger shear modulus for elastic problems with imposed shear stresses; (3) lower bounds on the full local stress for $\mu_1 = \mu_2$, which are seen to be optimal for a special class of imposed macroscopic stresses; (4) lower bounds on the local Von Mises equivalent stress, which are optimal for a similar special class of imposed macroscopic stress fields; and (5) lower bounds on the hydrostatic and deviatoric components of the local stress for the full set of imposed macroscopic stresses, subject to the hypotheses $\mu_1 = \mu_2$ or $\kappa_1 = \kappa_2$, respectively. These lower bounds are derived in section 5, and their attainability is demonstrated in section 6.

In what follows will adopt the notation $\kappa_+ = \max\{\kappa_1, \kappa_2\}$, $\mu_+ = \max\{\mu_1, \mu_2\}$, $\kappa_- = \min\{\kappa_1, \kappa_2\}$, and $\mu_- = \min\{\mu_1, \mu_2\}$.

### 3.1. Hydrostatic applied stress.

In this section we consider imposed macroscopic stresses that are hydrostatic, i.e., of the form $\overline{\sigma} = \overline{p}I$, where $\overline{p}$ is a constant and $I$ is the $d\times d$ identity matrix. Here it is assumed that the elastic materials inside the heterogeneous medium are well ordered; i.e., $(\mu_1 - \mu_2)(\kappa_1 - \kappa_2) > 0$. Without loss of generality we will suppose in this section that $\mu_1 > \mu_2$ and $\kappa_1 > \kappa_2$. We present lower bounds that are optimal for all imposed hydrostatic stresses. The configurations that attain the bounds are given by the Hashin–Shtrikman coated sphere and (cylinder) assemblages [15]. We describe the coated sphere assemblage made from a core of material one with a coating of material two and note that the coated cylinder assemblage is constructed similarly. We first fill $\mathbf{R}^3$ with an assemblage of spheres with sizes ranging down to the infinitesimal. Inside each sphere we place a smaller concentric sphere filled with "core" material one, and the surrounding coating is filled with material two. The volume fractions of materials one and two are taken to be the same for all of the coated spheres.

We begin by presenting optimal lower bounds on the moments of the local stress inside material one.

PROPOSITION 3.1 (optimal lower bounds on the local stress inside material one). *Consider any heterogeneous medium with volume (area) fractions of materials one and two given by $\theta_1$ and $\theta_2$. Then for an imposed hydrostatic macroscopic stress $\overline{\sigma} = \overline{p}I$ the local stress inside material one satisfies*

$$\langle \chi_1 |\sigma(\mathbf{x})|^r \rangle^{1/r} \geq \theta_1^{1/r} \frac{\sqrt{d}(\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu_2\kappa_1)}{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu_2\left(\theta_1\kappa_1 + \theta_2\kappa_2\right)}\,|\overline{p}| \qquad \text{for } 2 \leq r \leq \infty. \tag{3.1}$$

*Moreover, for $d = 2$ (3) and for every $r$ in $2 \leq r \leq \infty$ the lower bound is attained by the local stress inside the coated cylinder (sphere) assemblage with core of material one and coating of material two.*

A similar result holds for the local stress inside material two, as follows.

PROPOSITION 3.2 (optimal lower bounds on the local stress inside material two). *Consider any heterogeneous medium with volume (area) fractions of materials one and two given by $\theta_1$ and $\theta_2$. Then for an imposed hydrostatic macroscopic stress $\overline{\sigma} = \overline{p}I$ the local stress inside material two satisfies*

$$\langle \chi_2 |\sigma(\mathbf{x})|^r \rangle^{1/r} \geq \theta_2^{1/r} \frac{\sqrt{d}(\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu_1\kappa_2)}{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu_1\left(\theta_1\kappa_1 + \theta_2\kappa_2\right)}\,|\overline{p}| \qquad \text{for } 2 \leq r \leq \infty. \tag{3.2}$$

*Moreover, for $d = 2$ (3) and for every $r$ in $2 \leq r \leq \infty$ the lower bound is attained by the local stress inside the coated cylinder (sphere) assemblage with core of material two and coating of material one.*

The optimal lower bound on the $L^\infty$ norm of the magnitude of the local stress inside a random composite is given by the following.

PROPOSITION 3.3 (optimal lower bounds on the $L^\infty$ norm of the local stress). *Consider any heterogeneous medium with volume (area) fractions of materials one and two given by $\theta_1$ and $\theta_2$. Then for an imposed hydrostatic macroscopic stress $\overline{\sigma} = \overline{p}I$ the stress field inside the composite satisfies*

$$\| |\sigma(\mathbf{x})| \|_\infty \geq \frac{\sqrt{d}(\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu_2\kappa_1)}{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu_2\left(\theta_1\kappa_1 + \theta_2\kappa_2\right)}\,|\overline{p}|\,. \tag{3.3}$$

*Moreover, for $d = 2$ (3) the lower bound is attained by the local stress inside the coated cylinder (sphere) assemblage with core of material one and coating of material two.*

Arguments similar to those provided in section 5 deliver lower bounds on the local stress field when the two materials are not well ordered, i.e., $\mu_1 > \mu_2$ and $\kappa_1 < \kappa_2$. However, explicit calculation shows that the stress fields inside the coated sphere assemblage do not saturate the lower bounds for any combination of core and coating material when the materials are not well ordered.

**3.2. Deviatoric applied stress.** In this section we consider imposed macroscopic stresses that are purely deviatoric, i.e., $\overline{\sigma} = \overline{\sigma}^D$, where $\mathbf{\Pi}^H\overline{\sigma}^D = 0$. For two-dimensional elastic problems any deviatoric stress tensor can be expressed as the symmetric tensor product of two orthogonal unit vectors $\mathbf{a}$ and $\mathbf{b}$, i.e., $\overline{\sigma}^D = s(\mathbf{a}\odot\mathbf{b})$. Here $s$ is an arbitrary scalar. In three dimensions this type of stress tensor is referred to as a pure shear stress. For two-dimensional elastic problems we present lower bounds on the local stress and lower bounds on the local Von Mises equivalent stress that are optimal for all applied deviatoric stresses, and for three-dimensional problems we show that the lower bounds are optimal for any imposed pure shear stress. The bounds are attained by simple laminates made by alternately layering material one with material two in the proportions $\theta_1$ and $\theta_2$, respectively. The direction normal to the layers is denoted by $\mathbf{n}$. The optimal choice of layer direction is given by $\mathbf{n} = \mathbf{a}$ or $\mathbf{n} = \mathbf{b}$.

For a deviatoric macroscopic stress, we first present optimal lower bounds on the local stress inside the component material with the larger shear modulus. Here we denote the volume (area) fraction and indicator function of the material with the larger shear modulus by $\theta_+$ and $\chi_+$, respectively.

Proposition 3.4 (optimal lower bounds on the moments of the local stress inside the phase with larger shear modulus). *Consider any heterogeneous medium with area (volume) fractions of materials one and two given by $\theta_1$ and $\theta_2$. Then for an imposed deviatoric macroscopic stress $\overline{\sigma}^D$ the stress field inside the material with larger shear modulus satisfies*

$$\langle \chi_+ |\sigma(\mathbf{x})|^r \rangle^{1/r} \geq \theta_+^{1/r} \left|\overline{\sigma}^D\right| \qquad \textit{for } 2 \leq r \leq \infty. \tag{3.4}$$

*For $d = 2, 3$ and for every $2 \leq r \leq \infty$ when $\overline{\sigma}^D = s(\mathbf{a}\odot\mathbf{b})$ the lower bound* (3.4) *is attained by a simple laminate. The vector normal to the layer interface for the optimal laminate is chosen according to $\mathbf{n} = \mathbf{a}$ or $\mathbf{n} = \mathbf{b}$.*

The optimal lower bound on the $L^\infty$ norm of the magnitude of the local stress inside a random composite is given by the following.

Proposition 3.5 (optimal lower bounds on the $L^\infty$ norm of the local stress). *Consider any heterogeneous medium with area (volume) fractions of materials one and two given by $\theta_1$ and $\theta_2$. Then for an imposed deviatoric macroscopic stress $\overline{\sigma}^D$ the stress field inside the composite satisfies*

$$\| |\sigma(\mathbf{x})| \|_\infty \geq \left|\overline{\sigma}^D\right|. \tag{3.5}$$

*For $d = 2, 3$, when $\overline{\sigma}^D = s(\mathbf{a}\odot\mathbf{b})$ the lower bound* (3.5) *is attained by a simple laminate with $\mathbf{n} = \mathbf{a}$ or $\mathbf{n} = \mathbf{b}$.*

The next result provides a lower bound on the Von Mises equivalent stress inside the component material with the larger shear modulus.

Proposition 3.6 (optimal lower bounds on the moments of the local Von Mises equivalent stress inside the material with greater shear modulus). *Consider any heterogeneous medium with area (volume) fractions of materials one and two given by $\theta_1$ and $\theta_2$. Then for an imposed deviatoric macroscopic stress $\overline{\sigma}^D$ the local Von Mises stress field inside the material with larger shear modulus satisfies*

$$\left\langle \chi_+ |\mathbf{\Pi}^D \sigma(\mathbf{x})|^r \right\rangle^{1/r} \geq \theta_+^{1/r} \left|\overline{\sigma}^D\right| \qquad \textit{for } 2 \leq r \leq \infty. \tag{3.6}$$

*For $d = 2, 3$ and for every $2 \leq r \leq \infty$ when $\overline{\sigma}^D = s(\mathbf{a}\odot\mathbf{b})$ the lower bound* (3.6) *is attained by a simple laminate. The vector normal to the layer interface for the optimal laminate is chosen according to $\mathbf{n} = \mathbf{a}$ or $\mathbf{n} = \mathbf{b}$.*

The optimal lower bound on the $L^\infty$ norm of the the local Von Mises equivalent stress inside a random composite is given by the following.

PROPOSITION 3.7 (optimal lower bounds on the $L^\infty$ norm of the local Von Mises equivalent stress). *Consider any heterogeneous medium with area (volume) fractions of materials one and two given by $\theta_1$ and $\theta_2$. Then for an imposed deviatoric macroscopic stress $\overline{\sigma}^D$ the Von Mises equivalent stress field inside the composite satisfies*

$$\| |\mathbf{\Pi}^D \sigma(\mathbf{x})| \|_\infty \geq \left|\overline{\sigma}^D\right|. \tag{3.7}$$

*For $d = 2, 3$, when $\overline{\sigma}^D = s(\mathbf{a} \odot \mathbf{b})$ the lower bound* (3.7) *is attained by a simple laminate with $\mathbf{n} = \mathbf{a}$ or $\mathbf{n} = \mathbf{b}$.*

**3.3. Lower bounds on the local stress that are optimal for a special class of imposed macroscopic stress states.** In this section we start by considering heterogeneous materials made from two elastic materials sharing the same shear modulus, i.e., $\mu_1 = \mu_2 = \mu$. We present new lower bounds on the full local stress field that hold for every imposed macroscopic stress $\overline{\sigma}$. The lower bounds are shown to be optimal for special subsets $\mathcal{S}_1, \mathcal{S}_2$ of imposed macroscopic stresses. The subsets $\mathcal{S}_1, \mathcal{S}_2$ are given by the set of imposed constant stresses for which one can construct a confocal-ellipsoid (confocal-ellipse) assemblage that has a constant and purely hydrostatic stress and strain field inside the core phase. These sets can be described implicitly using necessary conditions of optimality as in [11, 10]. Here we use an explicit parameterization of this set recently developed in [32].

We describe the construction of a confocal-ellipsoid assemblage with a core of material one and a coating of material two, noting that the confocal-ellipse assemblage is constructed in a similar way. Consider $\mathbf{R}^3$ filled with an assemblage of ellipsoids with sizes ranging down to the infinitesimal. Here, all ellipsoids have the same shape and orientation of axes and differ only in their size. Inside each ellipsoid, one places a smaller confocal-ellipsoid filled with material one, and the surrounding coating is filled with material two. We call these coated ellipsoids. The part of $\mathbf{R}^3$ not covered by the coated ellipsoids has zero measure. The volume fractions of materials one and two are the same for each coated ellipsoid in the assemblage.

The set $\mathcal{S}_1$ of applied stresses is given explicitly by the parametric representation (see [32])

$$\overline{\sigma} = \left( \frac{\kappa_2(\kappa_1 + 2\frac{(d-1)\mu}{d})}{\kappa_1 - \kappa_2} + \frac{2\theta_1\mu(d-1)}{d} \right) I + 2\mu\theta_2 \left( M - \frac{1}{d} I \right), \tag{3.8}$$

where $M$ ranges over the totality of positive semidefinite $d \times d$ matrices with unit trace. For each $\overline{\sigma}$ in $\mathcal{S}_1$ one can construct a confocal-ellipsoid assemblage with a core of material one and a coating of material two such that the local stress inside the core is constant and hydrostatic. Here the axes of the ellipsoids correspond to the principal directions of $\overline{\sigma}$. The analogous parameterization of the set of imposed stresses for which the local stress is constant and hydrostatic for confocal ellipsoids with a core of material two is obtained by interchanging subscripts one and two in (3.8). This set of macroscopic stresses is denoted by $\mathcal{S}_2$.

The optimal lower bound on the moments of the local stress inside a random composite is given by the following.

PROPOSITION 3.8 (optimal lower bounds on the local stress inside material one for $\mu_1 = \mu_2$). *Consider any heterogeneous medium with volume (area) fractions of materials one and two given by $\theta_1$ and $\theta_2$. Then for any imposed macroscopic stress $\overline{\sigma}$ the stress field inside material one satisfies*

$$\langle \chi_1(\mathbf{x}) |\sigma(\mathbf{x})|^r \rangle^{1/r} \geq \theta_1^{1/r} \frac{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu\kappa_1}{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu(\theta_1\kappa_1 + \theta_2\kappa_2)} |\mathbf{\Pi}^H \overline{\sigma}| \qquad \text{for } 2 \leq r \leq \infty. \tag{3.9}$$

*Moreover, for $d = 2, 3$ and for every $r$ in $2 \le r \le \infty$, when $\overline{\sigma}$ lies in the set $\mathcal{S}_1$ the lower bound* (3.9) *is attained by the local stress inside material one for the confocal-ellipsoid (confocal-ellipse) assemblage associated with $\overline{\sigma}$.*

A similar result holds for local stress fields inside material two.

PROPOSITION 3.9 (optimal lower bounds on the local stress inside material two for $\mu_1 = \mu_2$). *Consider any heterogeneous medium with volume (area) fractions of materials one and two given by $\theta_1$ and $\theta_2$. Then for any imposed macroscopic stress field $\overline{\sigma}$ the stress field inside material two satisfies*

(3.10)
$$\langle \chi_2(\mathbf{x})|\sigma(\mathbf{x})|^r\rangle^{1/r} \ge \theta_2^{1/r} \frac{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu\kappa_2}{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu(\theta_1\kappa_1 + \theta_2\kappa_2)} |\mathbf{\Pi}^H\overline{\sigma}| \qquad \text{for } 2 \le r \le \infty.$$

*Moreover, for $d = 2, 3$ and for every $r$ in $2 \le r \le \infty$, when $\overline{\sigma}$ lies in the set $\mathcal{S}_2$, the lower bound* (3.10) *is attained by the local stress field inside material two for the confocal-ellipsoid (confocal-ellipse) assemblage with core of material two associated with $\overline{\sigma}$.*

We conclude this subsection by considering the two trivial lower bounds on the moments of the local Von Mises equivalent stress given by $\langle \chi_1(\mathbf{x})|\mathbf{\Pi}^D\sigma(\mathbf{x})|^r\rangle^{1/r} \ge 0$ and $\langle \chi_2(\mathbf{x})|\mathbf{\Pi}^D\sigma(\mathbf{x})|^r\rangle^{1/r} \ge 0$. In what follows we make no hypothesis on the bulk and shear moduli of the component materials and point out that the trivial bounds are optimal for two subsets of imposed stresses $\overline{\sigma}$. The subsets are denoted by $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{S}}_2$, and these sets correspond to the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ with $\mu = \mu_2$ and $\mu = \mu_1$, respectively.

PROPOSITION 3.10 (optimal lower bounds on the local Von Mises equivalent stress inside material one). *Consider any heterogeneous medium with volume (area) fractions of materials one and two given by $\theta_1$ and $\theta_2$. Then for any imposed macroscopic stress $\overline{\sigma}$ it is evident that the stress field inside material one satisfies*

$$\left\langle \chi_1(\mathbf{x})|\mathbf{\Pi}^D\sigma(\mathbf{x})|^r\right\rangle^{1/r} \ge 0 \qquad \text{for } 2 \le r \le \infty. \tag{3.11}$$

*Moreover, for $d = 2, 3$ and for every $r$ in $2 \le r \le \infty$, when $\overline{\sigma}$ lies in the set $\hat{\mathcal{S}}_1$ the lower bound* (3.11) *is attained by the local Von Mises stress inside material one for the confocal-ellipsoid (confocal-ellipse) assemblage associated with $\overline{\sigma}$.*

A similar result holds for local stress fields inside material two.

PROPOSITION 3.11 (optimal lower bounds on the local Von Mises equivalent stress inside material two). *Consider any heterogeneous medium with volume (area) fractions of materials one and two given by $\theta_1$ and $\theta_2$. Then for any imposed macroscopic stress field $\overline{\sigma}$ it is evident that the stress field inside material two satisfies*

$$\left\langle \chi_2(\mathbf{x})|\mathbf{\Pi}^D\sigma(\mathbf{x})|^r\right\rangle^{1/r} \ge 0 \qquad \text{for } 2 \le r \le \infty. \tag{3.12}$$

*Moreover, for $d = 2, 3$ and for every $r$ in $2 \le r \le \infty$, when $\overline{\sigma}$ lies in the set $\hat{\mathcal{S}}_2$, the lower bound* (3.12) *is attained by the local Von Mises stress field inside material two for the confocal-ellipsoid (confocal-ellipse) assemblage with core of material two associated with $\overline{\sigma}$.*

**3.4. Optimal lower bounds for general imposed macroscopic stresses and $\boldsymbol{\mu_1 = \mu_2}$.** In this section we consider two phase heterogeneous media subject to a general imposed macroscopic stress $\overline{\sigma}$. We suppose that the two materials share the same shear modulus $\mu = \mu_1 = \mu_2$, and we present optimal lower bounds on the hydrostatic part of the local stress.

The first result is a lower bound on all moments of the local hydrostatic stress inside each material.

PROPOSITION 3.12 (optimal lower bounds on the local hydrostatic stress with $\mu_1 = \mu_2$ for media subjected to a general imposed stress). *Consider any heterogeneous medium with volume (area) fractions of materials one and two given by $\theta_1$ and $\theta_2$. Then for any imposed macroscopic stress $\overline{\sigma}$ the hydrostatic component of the local stress field inside the ith material, $i = 1, 2$, satisfies*

(3.13)
$$\left\langle \chi_i |\mathbf{\Pi}^H \sigma(\mathbf{x})|^r \right\rangle^{1/r} \geq \theta_i^{1/r} \frac{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu\kappa_i}{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu(\theta_1\kappa_1 + \theta_2\kappa_2)} |\mathbf{\Pi}^H \overline{\sigma}| \qquad \text{for } 2 \leq r \leq \infty.$$

*Moreover, for $d = 2, 3$, the lower bound* (3.13) *is attained for every $r$ in $2 \leq r \leq \infty$ by the local hydrostatic stress field inside laminates made from layering the two materials in the prescribed proportions $\theta_1$ and $\theta_2$. Here the layering can be made along any direction* $\mathbf{n}$.

The next result provides a lower bound on the $L^\infty$ norm of the local stress inside the heterogeneous medium.

PROPOSITION 3.13 (optimal lower bounds on the $L^\infty$ norm of the local hydrostatic stress with $\mu_1 = \mu_2$ for media subjected to a general imposed stress). *Consider any heterogeneous medium with volume (area) fractions of materials one and two given by $\theta_1$ and $\theta_2$. Then for any imposed macroscopic stress $\overline{\sigma}$ the hydrostatic component of the local stress field satisfies*

$$\| |\mathbf{\Pi}^H \sigma(\mathbf{x})| \|_\infty \geq \frac{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu\kappa_+}{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu(\theta_1\kappa_1 + \theta_2\kappa_2)} |\mathbf{\Pi}^H \overline{\sigma}|. \tag{3.14}$$

*Moreover, for $d = 2, 3$, the lower bound* (3.14) *is attained by the local hydrostatic stress field inside a simply layered material. Here the layering can be made along any direction* $\mathbf{n}$.

**3.5. Optimal lower bounds for general imposed macroscopic stresses and $\boldsymbol{\kappa_1 = \kappa_2}$.** In this section we consider two phase heterogeneous media subject to any imposed macroscopic stress $\overline{\sigma}$. We suppose that the two materials share the same bulk modulus, i.e., $\kappa = \kappa_1 = \kappa_2$, and we present optimal lower bounds on the local Von Mises equivalent stress.

The first result is a lower bound on all moments of the local Von Mises equivalent stress inside the material with greater shear stiffness. To expedite the presentation we denote the indicator function of and proportion of the material with greater shear modulus by $\chi_+$ and $\theta_+$, respectively.

PROPOSITION 3.14 (optimal lower bounds on the moments of the local Von Mises equivalent stress inside the material with greater shear modulus for $\kappa_1 = \kappa_2$). *Consider any heterogeneous medium with volume (area) fractions of materials one and two given by $\theta_1$ and $\theta_2$. Then for any imposed macroscopic stress $\overline{\sigma}$ the local Von Mises stress field inside the material with larger shear modulus satisfies*

$$\left\langle \chi_+ |\mathbf{\Pi}^D \sigma(\mathbf{x})|^r \right\rangle^{1/r} \geq \theta_+^{1/r} \left|\mathbf{\Pi}^D \overline{\sigma}\right| \qquad \text{for } 2 \leq r \leq \infty. \tag{3.15}$$

*For $d = 2$ let $\psi_1, \psi_2$ be an orthonormal system of eigenvectors for $\overline{\sigma}$. Then for every $r$ in $2 \leq r \leq \infty$ the lower bound* (3.15) *is attained by the local Von Mises stress inside a simple laminate with layer normal $\mathbf{n} = \frac{\psi_1 + \psi_2}{\sqrt{2}}$. Here the deviatoric projection of the local stress inside this laminate is uniform and given by $\mathbf{\Pi}^D \sigma(\mathbf{x}) = \mathbf{\Pi}^D \overline{\sigma}$.*

*Remark.* For $d = 3$ a simple calculation based upon the explicit solution for the stress field inside a simple layered material given by (6.3)–(6.5) shows that the bound (3.15) is not attained by layered media.

The next result provides a lower bound on the $L^\infty$ norm of the local Von Mises equivalent stress inside the heterogeneous material.

PROPOSITION 3.15 (optimal lower bounds on the $L^\infty$ norm of the Von Mises equivalent stress for $\kappa_1 = \kappa_2$). *Consider any heterogeneous medium with volume (area) fractions of materials one and two given by $\theta_1$ and $\theta_2$. Then for any imposed macroscopic stress $\overline{\sigma}$ the local Von Mises equivalent stress inside the medium satisfies*

$$\| |\mathbf{\Pi}^D \sigma(\mathbf{x})| \|_\infty \geq \left|\mathbf{\Pi}^D \overline{\sigma}\right|. \tag{3.16}$$

*For $d = 2$, the lower bound* (3.16) *is attained by the local Von Mises stress inside a simple laminate with layer normal $\mathbf{n} = \frac{\psi_1+\psi_2}{\sqrt{2}}$.*

**4. Upper bounds on the macroscopic strength domain for random heterogeneous materials.** In this section we apply the optimal lower bounds on local stress fields to display new tight upper bounds for strength domains. We begin by considering the case of hydrostatic applied loads of the form $\overline{p}I$. For this case the local stress is of the form $\sigma(\mathbf{x}) = \overline{p}I + \hat{\sigma}(\mathbf{x})$ and $\langle\sigma\rangle = \overline{p}I$. The local stress is related to the local strain through (2.5) and satisfies the equations of elastic equilibrium specified in section 2.

In what follows we display an upper bound on the strength domain associated with norm of the local stress inside the composite. We suppose that failure is initiated inside phase one when $|\sigma(\mathbf{x})| = F_1$ over some subset of phase one, and inside phase two when $|\sigma(\mathbf{x})| = F_2$ over some subset of phase two. We suppose that only the volume fractions are known, i.e., $\langle\chi_1\rangle = \theta_1$ and $\langle\chi_2\rangle = 1 - \theta_1$, and we define the macroscopic strength domain $K^{Safe}$ to be the set of applied stresses $\overline{p}I$ for which the local stress field $\sigma(\mathbf{x})$ satisfies the local constraints

$$\chi_1(\mathbf{x})|\sigma(\mathbf{x})| < F_1, \qquad \chi_2(\mathbf{x})|\sigma(\mathbf{x})| < F_2. \tag{4.1}$$

We write

$$L_1(\theta_1) = \frac{\sqrt{3}(\kappa_1\kappa_2 + \frac{4}{3}\mu_2\kappa_1)}{\kappa_1\kappa_2 + \frac{4}{3}\mu_2(\theta_1\kappa_1 + \theta_2\kappa_2)} \quad \text{and} \quad L_2(\theta_1) = \frac{\sqrt{3}(\kappa_1\kappa_2 + \frac{4}{3}\mu_1\kappa_2)}{\kappa_1\kappa_2 + \frac{4}{3}\mu_1(\theta_1\kappa_1 + \theta_2\kappa_2)} \tag{4.2}$$

and define the upper bound $\overline{K}$ to be the set of matrices of the form $\overline{p}I$ that satisfy the constraints given by

$$|\overline{p}|L_1(\theta_1) \leq F_1 \quad \text{and} \quad |\overline{p}|L_2(\theta_1) \leq F_2. \tag{4.3}$$

We now present a tight upper bound on $K^{Safe}$.

PROPOSITION 4.1 (upper bound on the macroscopic strength domain for hydrostatic applied loads). *Suppose that $\mu_1 > \mu_2$, $\kappa_1 > \kappa_2$, $F_1 \leq F_2$, and $\theta_1$ is given; then $K^{Safe} \subset \overline{K}$. Moreover, $\overline{K}$ is a tight upper bound in that $\overline{p} \in \overline{K}$ implies that the local stress $|\sigma(\mathbf{x})|$ lies below the failure threshold inside both phases for the coated sphere construction with core material one and coating material two. And $\overline{p} \notin \overline{K}$ implies that the threshold has been exceeded everywhere inside the core phase of the coated sphere assemblage.*

*Proof.* Setting $r = \infty$ in (3.1) and (3.2) gives

$$|\overline{p}|L_1(\theta_1) \leq \|\chi_1|\sigma|\|_\infty \quad \text{and} \quad |\overline{p}|L_2(\theta_1) \leq \|\chi_2|\sigma|\|_\infty, \tag{4.4}$$

and inspection shows that $L_1(\theta_1) > 1 > L_2(\theta_1)$. Now for the coated sphere assemblage with a core phase of material one an easy computation shows that $|\overline{p}|L_1(\theta_1) = \|\chi_1|\sigma|\|_\infty = \||\sigma|\|_\infty$, and the upper bound follows. □

Next consider an applied deviatoric stress of the form $\overline{\sigma}^D = s(\mathbf{a} \odot \mathbf{b})$, where the unit vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal. For this case the local stress is of the form $\sigma(\mathbf{x}) = \overline{\sigma}^D + \hat{\sigma}(\mathbf{x})$ and $\langle\sigma\rangle = \overline{\sigma}^D$. The local stress is related to the local strain through (2.5) and satisfies the equations of elastic equilibrium specified in section 2. In this example no volume fraction constraints are imposed, and we consider the macroscopic strength domain $K^{Safe}$ defined to be the set of all applied stresses $\overline{\sigma}^D$ for which the local stress satisfies the local constraints given by

$$\chi_1(\mathbf{x})|\Pi^D\sigma(\mathbf{x})| < F_1 \quad \text{and} \quad \chi_2(\mathbf{x})|\Pi^D\sigma(\mathbf{x})| < F_2. \tag{4.5}$$

Now we define $\overline{K}$ to be the set of matrices of the form $\overline{\sigma}^D = s(\mathbf{a} \odot \mathbf{b})$ that satisfy the constraint given by

$$|\overline{\sigma}| \le \min(F_1, F_2) \tag{4.6}$$

and we have the following tight upper bound.

Proposition 4.2 (upper bound on the macroscopic strength domain for deviatoric applied loads). *Suppose $\mu_1 > \mu_2$, $\kappa_1 > \kappa_2$, and $F_1 \le F_2$; then $K^{Safe} \subset \overline{K}$. Moreover, $\overline{K}$ is a tight upper bound in that $\overline{\sigma}^D \in \overline{K}$ implies that the local deviatoric component of stress $|\Pi^D\sigma(\mathbf{x})|$ lies below the failure threshold inside both phases for a simple layered material with layer normal chosen parallel to $\mathbf{a}$ or $\mathbf{b}$. And $\overline{\sigma}^D \notin \overline{K}$ implies that the threshold has been exceeded everywhere inside phase one of the layered material.*

*Proof.* This result easily follows from (3.7) and (6.8). □

**5. Lower bounds on the local stress.** In this section, we derive the lower bounds on the local stress inside random heterogeneous media. The lower bounds are established with the aid of two inequalities that easily follow from Jensen's inequality. Let $\psi(\mathbf{x})$ be a $d \times d$ stress field defined on $\mathbf{R}^d$. Then

$$\langle\chi_i(\mathbf{x})\psi(\mathbf{x}) : \psi(\mathbf{x})\rangle \ \ \ge \frac{1}{\theta_i}\left|\langle\chi_i(\mathbf{x})\psi(\mathbf{x})\rangle\right|^2 \tag{5.1}$$

and

$$\langle\psi(\mathbf{x}) : \psi(\mathbf{x})\rangle \ \ \ge \left|\langle\psi(\mathbf{x})\rangle\right|^2. \tag{5.2}$$

These inequalities are strict in that equality holds in (5.1) only if $\psi(\mathbf{x})$ is constant on the set of points where $\chi_i = 1$, and in (5.2) only if $\psi(\mathbf{x})$ is constant everywhere.

**5.1. Hydrostatic applied stress.** In this section the imposed macroscopic stress is taken to be hydrostatic; i.e., $\overline{\sigma} = \overline{p}I$ and the two materials are well ordered. Without loss of generality we make the choice $\mu_1 > \mu_2$ and $\kappa_1 > \kappa_2$. The lower bounds (3.1), (3.2), (3.3) follow immediately from the optimal lower bounds on the hydrostatic component of local stress given in [28], on noting that $|\sigma(\mathbf{x})| \ge |\mathbf{\Pi}^H\sigma(\mathbf{x})|$. In section 6.1 we establish the optimality of these lower bounds for the well-ordered case.

**5.2. Deviatoric applied stress.** In this section we derive the lower bounds given by (3.4), (3.5), (3.6), and (3.7). Here we examine the local stress field inside the material with larger shear modulus, and without loss of generality we suppose that the shear modulus of material one is greater than that of material two; i.e., $\mu_1 > \mu_2$. In subsection 6.2 these lower bounds are shown to be optimal for imposed macroscopic deviatoric stresses in two dimensions and for imposed macroscopic stresses that are pure shear stresses in three dimensions.

We start by taking $\psi = \mathbf{\Pi}^D\sigma$ in (5.1) to obtain the basic lower bound given by

$$\left\langle \chi_1 \mathbf{\Pi}^D\sigma(\mathbf{x}) : \mathbf{\Pi}^D\sigma(\mathbf{x}) \right\rangle \;\; \geq \frac{1}{\theta_1}\left|\left\langle \chi_1 \mathbf{\Pi}^D\sigma(\mathbf{x}) \right\rangle\right|^2. \tag{5.3}$$

In what follows we obtain a lower bound for the right-hand side of (5.3). First we note that the average stress inside material one can be written as

$$\langle \chi_1\sigma(\mathbf{x})\rangle = \langle \chi_1 C(\mathbf{x})\epsilon(\mathbf{x})\rangle = C^1 \langle \chi_1\epsilon(\mathbf{x})\rangle. \tag{5.4}$$

Averaging the local stress-strain relation $\sigma(\mathbf{x}) = C(\mathbf{x})\epsilon(\mathbf{x})$ and applying the definition of the effective elastic tensor gives

$$\overline{\sigma} = C^e\overline{\epsilon} = C^2\overline{\epsilon} + (C^1 - C^2)\langle \chi_1\epsilon(\mathbf{x})\rangle, \tag{5.5}$$

the deviatoric part of the average macroscopic stress is given by

$$\mathbf{\Pi}^D\overline{\sigma} = 2\mu_2\mathbf{\Pi}^D\overline{\epsilon} + 2(\mu_1 - \mu_2)\left\langle \chi_1\mathbf{\Pi}^D\epsilon(\mathbf{x})\right\rangle, \tag{5.6}$$

and it follows that

$$\left\langle \chi_1\mathbf{\Pi}^D\sigma(\mathbf{x})\right\rangle = \frac{2\mu_1\mu_2}{\mu_1 - \mu_2}\left(\frac{1}{2\mu_2}\mathbf{\Pi}^D\overline{\sigma} - \mathbf{\Pi}^D(C^e)^{-1}\overline{\sigma}\right). \tag{5.7}$$

Up to this point we have assumed that the imposed macroscopic stress was given by an arbitrary $d \times d$ matrix. From now on in this subsection we will assume that the imposed macroscopic stress is deviatoric for both two- and three-dimensional elastic problems, i.e.,

$$\overline{\sigma} = \overline{\sigma}^D = \mathbf{\Pi}^D\overline{\sigma}^D, \tag{5.8}$$

and one obtains

$$\left\langle \chi_1\mathbf{\Pi}^D\sigma(\mathbf{x})\right\rangle = \frac{2\mu_1\mu_2}{\mu_1 - \mu_2}\left(\frac{1}{2\mu_2}\mathbf{\Pi}^D - \mathbf{\Pi}^D(C^e)^{-1}\right)\mathbf{\Pi}^D\overline{\sigma}. \tag{5.9}$$

We apply the Cauchy–Schwarz inequality to find that

$$\left|\left\langle \chi_1\mathbf{\Pi}^D\sigma(\mathbf{x})\right\rangle\right|^2 \geq \left(\frac{2\mu_1\mu_2}{\mu_1 - \mu_2}\right)^2 \frac{\left(\frac{1}{2\mu_2}\mathbf{\Pi}^D\overline{\sigma} : \mathbf{\Pi}^D\overline{\sigma} - ((C^e)^{-1}\mathbf{\Pi}^D\overline{\sigma} : \mathbf{\Pi}^D\overline{\sigma})^2\right)}{|\mathbf{\Pi}^D\overline{\sigma}|^2}. \tag{5.10}$$

The effective elasticity tensor $C^e$ satisfies the following well-known estimate (see [39]):

$$(C^e)^{-1}\overline{\sigma} : \overline{\sigma} \leq (\theta_1(C^1)^{-1} + \theta_2(C^2)^{-1})\overline{\sigma} : \overline{\sigma}. \tag{5.11}$$

From (5.11) one obtains

$$(C^e)^{-1}\mathbf{\Pi}^D\overline{\sigma} : \mathbf{\Pi}^D\overline{\sigma} \leq \left(\frac{\theta_1}{2\mu_1} + \frac{\theta_2}{2\mu_2}\right)\left|\mathbf{\Pi}^D\overline{\sigma}\right|^2, \tag{5.12}$$

and it follows from (5.12) that

$$\frac{1}{2\mu_2}\mathbf{\Pi}^D\overline{\sigma}:\mathbf{\Pi}^D\overline{\sigma}-(C^e)^{-1}\mathbf{\Pi}^D\overline{\sigma}:\mathbf{\Pi}^D\overline{\sigma}\;\geq\;\frac{\theta_1(\mu_1-\mu_2)}{2\mu_1\mu_2}\left|\mathbf{\Pi}^D\overline{\sigma}\right|^2. \tag{5.13}$$

From (5.3), (5.10), and (5.13), one obtains

$$\left\langle\chi_1|\mathbf{\Pi}^D\sigma(\mathbf{x})|^2\right\rangle\geq\theta_1\left|\mathbf{\Pi}^D\overline{\sigma}\right|^2. \tag{5.14}$$

For $p$ and $q$ such that $p\geq 1$ and $1/p+1/q=1$, Hölder's inequality gives

$$\theta_1^{1/q}\left\langle\chi_1|\mathbf{\Pi}^D\sigma(\mathbf{x})|^{2p}\right\rangle^{1/p}\geq\;\left\langle\chi_1|\mathbf{\Pi}^D\sigma(\mathbf{x})|^2\right\rangle \tag{5.15}$$

and hence the inequality

$$\left\langle\chi_1|\mathbf{\Pi}^D\sigma(\mathbf{x})|^{2p}\right\rangle^{1/p}\geq\theta_1^{1/p}\left|\mathbf{\Pi}^D\overline{\sigma}\right|^2, \tag{5.16}$$

for $1\leq p\leq\infty$, from which the bound (3.6) follows. The bound (3.4) now follows immediately from (5.16) and on noting that

$$\langle\chi_1|\sigma(\mathbf{x})|^r\rangle^{1/r}\;\geq\;\left\langle\chi_1|\mathbf{\Pi}^D\sigma(\mathbf{x})|^r\right\rangle^{1/r}\qquad\text{for } 2\leq r\leq\infty. \tag{5.17}$$

The bounds (3.5) and (3.7) follow immediately on substitution of $\psi(\mathbf{x})=\mathbf{\Pi}^D\sigma(\mathbf{x})$ into (5.2) and noting that

$$\||\sigma(\mathbf{x})|\|_\infty\geq\||\mathbf{\Pi}^D\sigma(\mathbf{x})|\|_\infty\geq\sqrt{\langle|\mathbf{\Pi}^D\sigma(\mathbf{x})|^2\rangle}\geq|\mathbf{\Pi}^D\overline{\sigma}|. \tag{5.18}$$

**5.3. Lower bounds on stress fields subject to general imposed macroscopic stresses and $\mu_1=\mu_2$.** In this subsection the imposed macroscopic stress is assumed to be any constant $d\times d$ stress tensor, $d=2,3$. In what follows we suppose that the two component materials share the same shear modulus, i.e., $\mu=\mu_1=\mu_2$, and we derive the lower bounds given by (3.9), (3.10), (3.13), and (3.14). In section 6 the lower bounds on the full local stress are shown to be optimal for special sets $\mathcal{S}_1$ and $\mathcal{S}_2$, and the lower bounds on the hydrostatic component of the local stress are shown to be optimal for all imposed macroscopic stresses.

It follows immediately from (5.4) and (5.5) that

$$\left\langle\chi_1\mathbf{\Pi}^H\sigma(\mathbf{x})\right\rangle=\frac{\kappa_1}{\kappa_1-\kappa_2}\left(\mathbf{\Pi}^H\overline{\sigma}-d\kappa_2\mathbf{\Pi}^H\overline{\epsilon}\right). \tag{5.19}$$

Taking $\psi=\mathbf{\Pi}^H\sigma$ in (5.1) shows that hydrostatic stress inside material one satisfies the following estimate:

$$\left\langle\chi_1\mathbf{\Pi}^H\sigma(\mathbf{x}):\sigma(\mathbf{x})\right\rangle\;\geq\frac{1}{\theta_1}\left|\left\langle\chi_1\mathbf{\Pi}^H\sigma(\mathbf{x})\right\rangle\right|^2. \tag{5.20}$$

For a composite consisting of two isotropic phases of equal shear moduli ($\mu_1=\mu_2=\mu$), Hill's relation [17] gives

$$C^e=2\mu\mathbf{\Pi}^D+d\kappa^e\mathbf{\Pi}^H, \tag{5.21}$$

where

$$\kappa^e=(\theta_1\kappa_1+\theta_2\kappa_2)-\frac{\theta_1\theta_2(\kappa_1-\kappa_2)^2}{\theta_1\kappa_2+\theta_2\kappa_1+2\frac{d-1}{d}\mu}. \tag{5.22}$$

From (2.9) and (5.21), one obtains

$$\mathbf{\Pi}^H\overline{\epsilon} = \frac{1}{d\kappa^e}\mathbf{\Pi}^H\overline{\sigma}, \tag{5.23}$$

and hence

$$\left\langle \chi_1 \mathbf{\Pi}^H \sigma(\mathbf{x}) \right\rangle = \frac{\kappa_1}{\kappa_1 - \kappa_2}\left(1 - \frac{\kappa_2}{\kappa^e}\right)\mathbf{\Pi}^H\overline{\sigma}. \tag{5.24}$$

From estimate (5.20) we recover

$$\left\langle \chi_1 \mathbf{\Pi}^H \sigma(\mathbf{x}) : \sigma(\mathbf{x}) \right\rangle \geq \frac{\kappa_1^2}{\theta_1(\kappa_1 - \kappa_2)^2}\left(1 - \frac{\kappa_2}{\kappa^e}\right)^2 \left|\mathbf{\Pi}^H\overline{\sigma}\right|^2, \tag{5.25}$$

and using the formula for $\kappa^e$ given (5.22), we express (5.25) as

$$\left\langle \chi_1 \mathbf{\Pi}^H \sigma(\mathbf{x}) : \sigma(\mathbf{x}) \right\rangle \geq \theta_1 \left(\frac{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu\kappa_1}{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu(\theta_1\kappa_1 + \theta_2\kappa_2)}\right)^2 \left|\mathbf{\Pi}^H\overline{\sigma}\right|^2. \tag{5.26}$$

An application of Hölder's inequality to (5.26) delivers

$$\left\langle \chi_1 |\mathbf{\Pi}^H \sigma(\mathbf{x})|^r \right\rangle^{1/r} \;\geq\; \theta_1^{1/r} \frac{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu\kappa_1}{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu(\theta_1\kappa_1 + \theta_2\kappa_2)} \left|\mathbf{\Pi}^H\overline{\sigma}\right| \tag{5.27}$$

for $2 \leq r \leq \infty$. Identical arguments give lower bounds on the hydrostatic stress inside phase two, and the lower bound (3.13) is established. The $L^\infty$ bound, (3.14), follows from the bound (3.13) by taking $r = \infty$, noting that $\||\mathbf{\Pi}^H\sigma(\mathbf{x})|\|_\infty \geq \|\chi_i|\mathbf{\Pi}^H\sigma(\mathbf{x})|\|_\infty$ for $i = 1, 2$.

To establish the bounds (3.9) and (3.10), we observe that because of orthogonality one obtains

$$|\sigma(\mathbf{x})|^2 = |\mathbf{\Pi}^H\sigma(\mathbf{x})|^2 + |\mathbf{\Pi}^D\sigma(\mathbf{x})|^2 \;\geq\; |\mathbf{\Pi}^H\sigma(\mathbf{x})|^2, \tag{5.28}$$

and hence

$$\langle \chi_i(\mathbf{x})|\sigma(\mathbf{x})|^r\rangle^{1/r} \;\geq\; \left\langle \chi_i(x)|\mathbf{\Pi}^H\sigma(\mathbf{x})|^r\right\rangle^{1/r}. \tag{5.29}$$

The bounds (3.9) and (3.10) follow from (3.13) and (5.29).

**5.4. Lower bounds on stress fields subject to general imposed macroscopic stresses and $\boldsymbol{\kappa_1 = \kappa_2}$.** In this subsection no constraints are placed on the imposed macroscopic stress. The imposed macroscopic stress can be any constant $d \times d$ stress tensor, $d = 2, 3$. In what follows we suppose that the two component materials share the same bulk modulus, i.e., $\kappa = \kappa_1 = \kappa_2$, and we derive new lower bounds on the local Von Mises stress inside the material with greater shear stiffness. To fix ideas we suppose that material one has the greater shear stiffness, i.e., $\mu_1 > \mu_2$. We will establish the lower bound (3.15) with the aid of the following observation, whose proof is provided in the appendix.

*Form of the effective stiffness tensor for mixtures of two elastically isotropic materials having common bulk modulus.* For $\kappa = \kappa_1 = \kappa_2$, the effective elasticity tensor $C^e$ can be written as

$$C^e = \mathbf{\Pi}^D C^e \mathbf{\Pi}^D + d\kappa\mathbf{\Pi}^H, \tag{5.30}$$

and consequently

$$(C^e)^{-1} = (\mathbf{\Pi}^D C^e \mathbf{\Pi}^D)^{-1} + \frac{1}{d\kappa}\mathbf{\Pi}^H. \tag{5.31}$$

Choosing $\psi = \mathbf{\Pi}^D\sigma$ in (5.1) gives

$$\left\langle \chi_1 \mathbf{\Pi}^D \sigma(\mathbf{x}) : \sigma(\mathbf{x}) \right\rangle \ \geq \frac{1}{\theta_1} \left| \left\langle \chi_1 \mathbf{\Pi}^D \sigma(\mathbf{x}) \right\rangle \right|^2. \tag{5.32}$$

We notice from (5.30) that $C^e$ commutes with $\mathbf{\Pi}^D$, which implies that $(C^e)^{-1}$ commutes with $\mathbf{\Pi}^D$. Thus from (2.9) it follows that

$$\mathbf{\Pi}^D\overline{\epsilon} = \mathbf{\Pi}^D (C^e)^{-1}\overline{\sigma} = (C^e)^{-1}\mathbf{\Pi}^D\overline{\sigma}. \tag{5.33}$$

Thus (5.7) becomes

$$\left\langle \chi_1 \mathbf{\Pi}^D \sigma(\mathbf{x}) \right\rangle = \frac{2\mu_1\mu_2}{\mu_1 - \mu_2}\left( \frac{1}{2\mu_2}\mathbf{\Pi}^D\overline{\sigma} - (C^e)^{-1}\mathbf{\Pi}^D\overline{\sigma} \right). \tag{5.34}$$

We apply the Cauchy–Schwarz inequality to find that

$$\left| \left\langle \chi_1 \mathbf{\Pi}^D \sigma(\mathbf{x}) \right\rangle \right|^2 \geq \left( \frac{2\mu_1\mu_2}{\mu_1 - \mu_2} \right)^2 \frac{\left(\frac{1}{2\mu_2}\mathbf{\Pi}^D\overline{\sigma} : \mathbf{\Pi}^D\overline{\sigma} - (C^e)^{-1}\mathbf{\Pi}^D\overline{\sigma} : \mathbf{\Pi}^D\overline{\sigma}\right)^2}{|\mathbf{\Pi}^D\overline{\sigma}|^2}. \tag{5.35}$$

With (5.35) in hand, we proceed as in subsection 5.2 to discover

$$\left\langle \chi_1 \mathbf{\Pi}^D \sigma(\mathbf{x}) : \sigma(\mathbf{x}) \right\rangle \ \geq \ \theta_1 \left| \mathbf{\Pi}^D\overline{\sigma} \right|^2. \tag{5.36}$$

The bounds (3.15) now follow from Hölder's inequality and arguments identical to those of subsection 5.2.

The bound (3.16) follows directly from

$$\| |\mathbf{\Pi}^D\sigma(\mathbf{x})| \|_\infty \ \geq \ \sqrt{\langle \mathbf{\Pi}^D\sigma(\mathbf{x}) : \sigma(\mathbf{x}) \rangle} \ \geq \ \left| \mathbf{\Pi}^D\overline{\sigma} \right|. \tag{5.37}$$

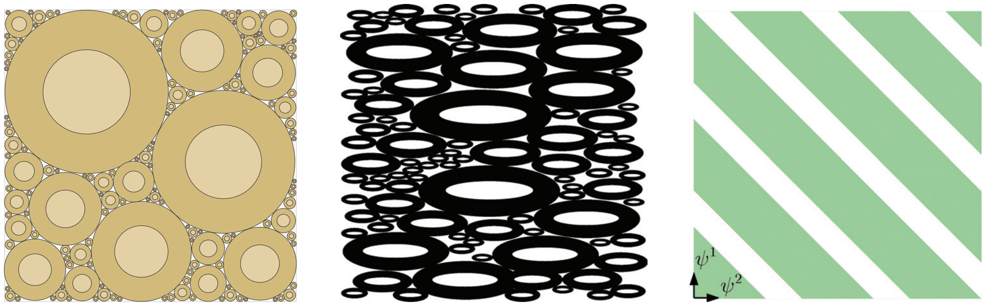
## 6. Microstructures that support optimal local fields.

It is well known that the coated sphere, coated ellipsoid, and laminated microstructures possess optimal effective elastic properties; for reviews of the literature, see [32] and [46]. In the following subsections we show that these microstructures possess optimal local field properties as well.

### 6.1. The coated sphere construction and optimal lower bounds on local stress fields.

In this section, it is shown that the lower bounds presented in subsection 3.1 are attained by the stress fields inside the Hashin–Shtrikman [14, 15] coated cylinder and sphere assemblages; see Figure 6.1(a). We introduce the normalized $L^p$ norm of a field $f$ over a domain $S$ by $(|S|^{-1}\int_S |f(\mathbf{x})|^p \, d\mathbf{x})^{1/p}$. One striking feature of the fields inside the coated sphere and cylinder assemblage is that the normalized $L^p$ norm of the local stress or strain taken over a prototypical coated cylinder or sphere is the same as the $L^p$ norm of the whole assemblage. Thus the $L^p$ norms of local fields inside these assemblages are obtained by computing the $L^p$ norm of a prototypical coated sphere or disk.

Assume that the applied field $\overline{\sigma}$ is hydrostatic, $\overline{\sigma} = \overline{p}I$, and consider the stress field inside a prototypical coated sphere (cylinder) centered at the origin with core of



(a) Coated cylinder assemblage (b) Confocal-ellipse assemblage (c) Layered material

FIG. 6.1. *Extremal microstructures.*

material one and coating of material two. We recall from [28] that the lower bound on the hydrostatic component of stress inside material one is given by the right-hand side of (3.1) and is attained by the stress field inside the core phase of the coated sphere (cylinder) assemblage.

Therefore it follows that the local stress inside material one attains the optimal lower bound on the hydrostatic stress given in [28], and attainability of the lower bound (3.1) follows. Similar arguments show that the lower bound (3.2) is attained by the stress field inside material two of the coated sphere (cylinder) assemblage with core of material two and coating of material one.

Next we show that the $L^\infty$ bound (3.3) is attained by the stress field inside the coated sphere (cylinder) assemblage with core material one and coating material two. A straightforward calculation shows that

$$\begin{aligned}\|\chi_1|\sigma|\|_\infty^2 - \|\chi_2|\sigma|\|_\infty^2 &= \frac{d}{\left(\kappa_1\kappa_2 + 2\left(\frac{d-1}{d}\right)\mu_2\left(\theta_1\kappa_1+\theta_2\kappa_2\right)\right)^2}\,|\overline{p}|^2\\ &\quad\times(\kappa_1-\kappa_2)\left(4\mu_2\left(\frac{d-1}{d^2}\right)(d\kappa_1\kappa_2+\mu_2((d-2)\kappa_1+d\kappa_2))\right),\end{aligned} \tag{6.1}$$

and it is evident from (6.1) that $\|\chi_1|\sigma|\|_\infty^2 \geq \|\chi_2|\sigma|\|_\infty^2$. Hence

$$\||\sigma|\|_\infty = \|\chi_1|\sigma|\|_\infty = \frac{\sqrt{d}(\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu_2\kappa_1)}{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu_2\left(\theta_1\kappa_1+\theta_2\kappa_2\right)}\,|\overline{p}|, \tag{6.2}$$

and the lower bound (3.3) is attained.

**6.2. The stress field inside simple laminates and optimal bounds on local fields.** For a laminate made from two isotropic phases the local stress field is piecewise constant under uniform applied stress $\overline{\sigma}$. The (constant) field inside the $i$th phase is denoted by $\overline{\sigma}^i$, and calculation gives

$$\overline{\sigma}^1 = \left((C^1)^{-1} + \frac{\theta_1}{\theta_2}(C^2)^{-1}\right)^{-1}\left(\lambda\odot\mathbf{n} + \frac{1}{\theta_2}(C^2)^{-1}\overline{\sigma}\right), \tag{6.3}$$

$$\overline{\sigma}^2 = \left((C^2)^{-1} + \frac{\theta_2}{\theta_1}(C^1)^{-1}\right)^{-1}\left(-\lambda\odot\mathbf{n} + \frac{1}{\theta_1}(C^1)^{-1}\overline{\sigma}\right), \tag{6.4}$$

with

$$\lambda \odot \mathbf{n} = -A(\overline{\sigma}\mathbf{n} \odot \mathbf{n}) \; + \left( B(\overline{\sigma}\mathbf{n} \cdot \mathbf{n}) + C\frac{\operatorname{tr}\overline{\sigma}}{d} \right) \mathbf{n} \odot \mathbf{n}, \tag{6.5}$$

where

$$\begin{aligned} A &= \frac{\Delta\mu}{\mu_1\mu_2}, \\ B &= \Delta\mu \left( \frac{\langle\kappa\rangle\,(1-\frac{2}{d}) + \langle\mu\rangle\,\frac{\kappa_1\kappa_2}{\mu_1\mu_2}}{2\mu_1\mu_2\,\langle\kappa\rangle\,(1-\frac{1}{d}) + \kappa_1\kappa_2\,\langle\mu\rangle} \right), \\ C &= \frac{\Delta\mu\,\langle\kappa\rangle - \Delta\kappa\,\langle\mu\rangle}{2\mu_1\mu_2\,\langle\kappa\rangle\,(1-\frac{1}{d}) + \kappa_1\kappa_2\,\langle\mu\rangle}, \end{aligned} \tag{6.6}$$

where $\langle\tilde{\mu}\rangle = \theta_1\mu_2 + \theta_2\mu_1$ and $\langle\tilde{\kappa}\rangle = \theta_1\kappa_2 + \theta_2\kappa_1$. Here $\Delta\mu = \mu_1 - \mu_2$, $\Delta\kappa = \kappa_1 - \kappa_2$, $\langle\mu\rangle = \theta_1\mu_1 + \theta_2\mu_2$, and $\langle\kappa\rangle = \theta_1\kappa_1 + \theta_2\kappa_2$.

We recall that both deviatoric applied stress in two dimensions as well as pure shear stresses in three dimensions can be expressed in the form $\overline{\sigma} = s(\mathbf{a} \odot \mathbf{b})$ with $\mathbf{a} \cdot \mathbf{b} = 0$, $|\mathbf{a}| = 1$, and $|\mathbf{b}| = 1$. On choosing $\mathbf{n} = \mathbf{a}$ or $\mathbf{n} = \mathbf{b}$ in (6.5), one easily sees that

$$\lambda \odot \mathbf{n} = -\frac{\Delta\mu}{2\mu_1\mu_2}\,\overline{\sigma}, \tag{6.7}$$

and it follows from (6.3) and (6.4) that

$$\overline{\sigma}^1 = \overline{\sigma}^2 = \overline{\sigma}. \tag{6.8}$$

From this observation it is evident that the stress field inside this simple laminate attains the bounds (3.4), (3.5), (3.6), and (3.7).

When both materials share the same shear modulus we find that the hydrostatic stress fields inside simple laminates have extremal properties. We demonstrate first that the lower bounds (3.13) and (3.14) are attained by the hydrostatic stress fields inside any simple laminate. For a simple laminate the stress field inside each material is constant, and hence both sides of inequality (5.20) are in fact equal and

$$\left\langle \chi_1 \mathbf{\Pi}^H \sigma(\mathbf{x}) : \sigma(\mathbf{x}) \right\rangle = \frac{1}{\theta_1} \left| \left\langle \chi_1 \mathbf{\Pi}^H \sigma(\mathbf{x}) \right\rangle \right|^2 = \theta_1 \left| \mathbf{\Pi}^H \overline{\sigma}^1 \right|^2, \tag{6.9}$$

where $\overline{\sigma}^1$ is the constant field inside material one. On the other hand, since $\mu_1 = \mu_2$ one obtains from (5.24) and (5.22) that

$$\frac{1}{\theta_1} \left| \left\langle \chi_1 \mathbf{\Pi}^H \sigma(\mathbf{x}) \right\rangle \right|^2 = \theta_1 \left( \frac{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu\kappa_1}{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu(\theta_1\kappa_1 + \theta_2\kappa_2)} \right)^2 \left| \mathbf{\Pi}^H \overline{\sigma} \right|^2. \tag{6.10}$$

Thus it follows from (6.9) and (6.10) that the local hydrostatic stress inside a simply layered laminate attains the bound (3.13) when $i = 1$. Given $\mu_1 = \mu_2$, these arguments show that if the stress field is constant inside material one, then its hydrostatic part attains the lower bound (3.13). Similar arguments show the optimality of the bound (3.13) when $i = 2$. The fact that the hydrostatic stress inside a rank one laminate attains the bound (3.13) for $i = 1$ and $i = 2$ implies that it also attains the $L^\infty$ bound (3.14).

We suppose that $\kappa_1 = \kappa_2$, $d = 2$, and we denote the orthonormal system of eigenvectors for a prescribed $2 \times 2$ imposed macroscopic stress by $\psi^1, \psi^2$. We show that the lower bounds presented in subsection 3.5 are attained by the stress fields inside a rank one laminate with layering direction $\mathbf{n} = \frac{1}{\sqrt{2}}(\psi^1 + \psi^2)$; see Figure 6.1(c). Choosing $\kappa_1 = \kappa_2$ and $\mathbf{n} = \frac{1}{\sqrt{2}}(\psi^1 + \psi^2)$ in (6.5) gives

$$\lambda \odot \mathbf{n} = -\frac{\Delta\mu}{2\mu_1\mu_2}\,\mathbf{\Pi}^D\overline{\sigma}, \tag{6.11}$$

and it follows from (6.3) and (6.4) that

$$\mathbf{\Pi}^D\overline{\sigma}^1 = \mathbf{\Pi}^D\overline{\sigma}^2 = \mathbf{\Pi}^D\overline{\sigma}. \tag{6.12}$$

From this observation it is evident that the stress field inside this rank one laminate attains the bounds (3.15) and (3.16).

**6.3. The confocal ellipsoid (ellipse) assemblage and optimal lower bounds on local stress fields for subsets of imposed macroscopic loads.** In this section, it is shown that the lower bounds (3.9), (3.10), (3.11), and (3.12) are attained by the stress fields inside the confocal-ellipsoid and confocal-ellipse assemblages; see Figure 6.1(b). Assuming that the uniform stress lies in $\mathcal{S}_1$, it follows that there is a confocal-ellipsoid (confocal-ellipse) assemblage with core of material one and coating of material two associated with $\overline{\sigma}$ such that the local stress inside the core material is constant and hydrostatic. Since the stress field inside material one is constant, it follows from earlier arguments that

$$\left\langle \chi_1 \mathbf{\Pi}^H\sigma(\mathbf{x}) : \sigma(\mathbf{x}) \right\rangle = \theta_1 \left( \frac{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu\kappa_1}{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu(\theta_1\kappa_1 + \theta_2\kappa_2)} \right)^2 \left|\mathbf{\Pi}^H\overline{\sigma}\right|^2. \tag{6.13}$$

On the other hand, since the stress field in material one is hydrostatic, one sees that

$$\left\langle \chi_1 \mathbf{\Pi}^D\sigma(\mathbf{x}) : \sigma(\mathbf{x}) \right\rangle = 0, \tag{6.14}$$

and it is also evident that the lower bounds (3.11) are attained. From (6.13) and (6.14) and the fact that $\sigma(\mathbf{x}) = \mathbf{\Pi}^H\sigma(\mathbf{x}) + \mathbf{\Pi}^D\sigma(\mathbf{x})$, one obtains

$$\langle \chi_1 \sigma(\mathbf{x}) : \sigma(\mathbf{x}) \rangle = \theta_1 \left( \frac{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu\kappa_1}{\kappa_1\kappa_2 + 2\frac{d-1}{d}\mu(\theta_1\kappa_1 + \theta_2\kappa_2)} \right)^2 \left|\mathbf{\Pi}^H\overline{\sigma}\right|^2, \tag{6.15}$$

from which optimality of the bound (3.9) follows. Identical arguments show that the local stress field inside material two of a confocal-ellipsoid (confocal-ellipse) assemblage with core of material two and coating of material one saturates the bounds (3.10) and (3.12).

**7. Conclusion.** The results presented in this work are partial. They are given by a set of optimal bounds that apply to several different types of load cases and (or) for specific constraints on the elastic properties of the constituent materials. Naturally the ultimate goal is to find optimal lower bounds on local stress fields for all imposed macroscopic stresses and for arbitrary choices of material properties. In this section we briefly review our methodology to identify the issues that prevent us from obtaining results for more general situations.

The method developed here identifies the appropriate applications of Jensen's inequality and Hölder's inequality necessary to bound the moments of the local field inside each material phase. A slight reinterpretation of (5.3) and (5.7), (5.19) and (5.20), as well as (5.32) and (5.34) of section 5 shows that this procedure delivers lower bounds given in terms of quadratic functions of the averaged stress and strain fields. For average stresses that are either hydrostatic or of pure shear type we have identified cases when these lower bounds can be expressed as functions of the overall strain or compliance energy of the composite. Alternatively, when the effective elastic tensor for the composite can be written as a sum of projections onto the subspaces of deviatoric and hydrostatic strains (via the exact relations of Hill [17] or (5.30) developed in the appendix), we have identified cases when the lower bounds on the local fluctuations can also be expressed as functions of the overall strain or compliance energy. For these cases we can apply the known bounds on overall elastic energies for composite materials to obtain lower bounds on the moments of the local fields given in terms of the volume fractions and the elastic constants of the constituent materials.

In order to extend the results presented here we need to further develop our understanding of the set of average stress strain pairs associated with composite media. One direction is to pursue the derivation of new explicit lower bounds on the moments of local fields given in terms of material properties and volume fractions for the general contractions of average stress and strain pairs appearing in (5.3) and (5.7), (5.19) and (5.20), as well as (5.32) and (5.34). In this context we point out the potential applicability of the recent methods developed for bounding the set of average stress and strain pairs presented in [44] and for bounding the pairs of average current and electric fields for composites made from nonlinear conducting materials [34].

A second related issue concerns the optimality of lower bounds on the moments of the local stress field. We point out here that Jensen's inequality and convexity show directly that the lower bounds delivered by combining (5.3) and (5.7), (5.19) and (5.20), as well as (5.32) and (5.34) of section 5 are attained exclusively by microstructures supporting constant local stress fields inside the phase of interest. In this work we have utilized the *constant field* microstructures given by coated spheres, ellipsoids, and rank one laminates. Other microgeometries supporting constant local fields include the "Vigdergauz" microstructures [48] and the more recently discovered "E" inclusions [30]. Thus future progress is contingent on the ability to characterize the range of average stress and strain pairs generated by the *constant field* microstructures.

We conclude by noting that in this treatment the analysis is carried out for situations where the underlying probability measure describing the local elastic properties of the random medium is not known. In this context we consider cases when only the material properties and volume fraction information are available. When the probability measure as well as other pieces of information describing the composite are available, it is possible to say more. For example, in [24] the authors consider elastic–perfectly plastic composites described by partitions of the material body into subdomains where the random elastic property inside each subdomain is described by an independent and identically distributed random variable. In that work the local set undergoing plastic deformation is shown to be of a fractal nature, and the overall transition from elastic to plastic is smooth. This may be compared with the extremal layered materials described in Proposition 4.2. When the magnitude of the imposed shear stress $\overline{\sigma} = s(\mathbf{a} \odot \mathbf{b})$ equals the plastic limit of the stiff material, Proposition 4.2 shows that for layerings of two materials with layer normal oriented along either the $\mathbf{a}$ or $\mathbf{b}$ axis the local Von Mises stress attains the plastic limit everywhere inside the stiffer layers. When the magnitude of the imposed shear stress lies below the plastic limit of the stiff material, Proposition 4.2 states that the local Von Mises stress lies below the plastic limit everywhere inside the layered composite. Thus the overall elastic to plastic transition for these layered composite geometries is not smooth but instead is identical to the abrupt elastic to plastic transition of the stiff layer.

**Appendix.** Here we provide a proof for (5.30) presented in section 5.4.

Let $\overline{\epsilon} = \langle \epsilon \rangle$. Then since the two materials are isotropic and $\kappa_1 = \kappa_2 = \kappa$, one obtains

$$\begin{aligned} C^e \overline{\epsilon} &= \langle C(\mathbf{x})\, \epsilon(\mathbf{x}) \rangle \\ &= \left\langle 2\mu(\mathbf{x}) \mathbf{\Pi}^D \epsilon(\mathbf{x}) \right\rangle + \left\langle d\, \kappa\, \mathbf{\Pi}^H \epsilon(\mathbf{x}) \right\rangle \\ &= \mathbf{\Pi}^D \left\langle 2\mu(\mathbf{x}) \epsilon(\mathbf{x}) \right\rangle + d\, \kappa\, \mathbf{\Pi}^H \overline{\epsilon}. \end{aligned} \tag{A.1}$$

Since $\mathbf{\Pi}^H \mathbf{\Pi}^D = 0$, one obtains from (A.1) that

$$\mathbf{\Pi}^H C^e \overline{\epsilon} = d\, \kappa\, \mathbf{\Pi}^H \overline{\epsilon}. \tag{A.2}$$

For a deviatoric uniform field $\overline{\epsilon} = \mathbf{\Pi}^D \overline{\epsilon}$, it follows from (A.1) that

$$C^e \mathbf{\Pi}^D \overline{\epsilon} = \mathbf{\Pi}^D \left\langle 2\mu(\mathbf{x}) \epsilon(\mathbf{x}) \right\rangle. \tag{A.3}$$

Thus for any two uniform strain fields $\xi$ and $\eta$

$$C^e \mathbf{\Pi}^H \eta : \mathbf{\Pi}^D \xi = C^e \mathbf{\Pi}^D \xi : \mathbf{\Pi}^H \eta = \mathbf{\Pi}^H C^e \mathbf{\Pi}^D \xi : \eta = 0, \tag{A.4}$$

and using this observation, one finds that

$$\begin{aligned} C^e \xi : \eta &= C^e (\mathbf{\Pi}^D \xi + \mathbf{\Pi}^H \xi) : (\mathbf{\Pi}^D \eta + \mathbf{\Pi}^H \eta) \\ &= C^e \mathbf{\Pi}^D \xi : \mathbf{\Pi}^D \eta + C^e \mathbf{\Pi}^H \xi : \mathbf{\Pi}^H \eta \\ &= \mathbf{\Pi}^D C^e \mathbf{\Pi}^D \xi : \eta + \mathbf{\Pi}^H C^e \mathbf{\Pi}^H \xi : \eta. \end{aligned} \tag{A.5}$$

From (A.2) one obtains

$$\mathbf{\Pi}^H C^e \mathbf{\Pi}^H \xi : \eta = d\, \kappa\, \mathbf{\Pi}^H \xi : \eta. \tag{A.6}$$

Thus (A.5) becomes

$$C^e \xi : \eta = (\mathbf{\Pi}^D C^e \mathbf{\Pi}^D + d\, \kappa\, \mathbf{\Pi}^H)\, \xi : \eta, \tag{A.7}$$

from which (5.30) follows.

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